Performance Analysis of Discrete-Time Average Consensus under Uniform Constant Time Delays

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Abstract: In this paper, considering discrete-time average consensus with uniform constant time delays, we focus on the stability, the final consensus value and the convergence rate. Specifically, it is proven that average consensus is robust to time delays via a matrix theory-based approach. Then, the deviation of the final value from the average of the initial states is expressed explicitly. It is found that average consensus is only preserved in some special topologies, i.e., regular graphs. Finally, for the regular graph, it is proven that the convergence rate of average consensus decreases with time delays by comparing the second largest eigenvalues modulus (SLEM) of the update matrices.

Keywords: Average consensus, uniform time delays, convergence rate, stability, final value.

1. INTRODUCTION

Average consensus is a highly efficient computing and control distributed algorithm through which states of all nodes in a network could reach to the average of their initial values regarding a specific state via local interactions. It has been extensively studied with respect to both theory and applications in the past few decades (Chen et al. [2013], Olfati-Saber et al. [2007], Qin et al. [2014]). Typical applications of average consensus have been developed in the area of formation control (Fax and Murray [2004]), coordination and cooperation (Ren et al. [2005], Jadbabaie and J. Lin [2003]), flocking (Olfati-Saber [2006]), time synchronization (He et al. [2014a], Carli and Zampieri [2014], He et al. [2014b]), data fusion (Xiao et al. [2005]), energy management (Zhao et al. [2016]), etc.

To reach average consensus, neighboring nodes need to communicate with each other and exchange their states in each iteration of the algorithms. However, the communications may be unreliable especially for wireless channels. The delayed packets influence the performance of average consensus regarding the stability, the final value and the convergence rate. Numerous efforts have been devoted to studying different consensus algorithms with communication time delays.

For continuous-time model, Olfati-saber et al. in (Olfati-Saber and Murray [2004]) consider the uniform constant time delays and prove that the average consensus problem is asymptotically solved if time delays is within a constant upper bound. This work is extended in (Bliman and Ferrari-Trecate [2008]), where sufficient and/or necessary conditions for stability are obtained by considering more general delay model, i.e., constant or time-varying, uniform or nonuniform delays. Meanwhile, much attention has been paid to other continuous-time consensus problems under time delays, including the upper bound analysis of delays to achieve the stability (Wang et al. [2013]), the non-linear dynamics with heterogeneous delays (Papachristodoulou et al. [2010]), and the output of continuous-time consensus with heterogeneous feedback delays (Maz et al. [2010]), etc.

For discrete-time consensus model, Wang et al. introduce a “pre-leader-follower” decomposition method to address consensus stability problem under time-invariant and time-varying delays and conditions regarding the underlying graph are given to guarantee stability in (Xiao and Wang [2006]). A class of consensus protocols are proposed in (Xiao and Wang [2008]) to realize consensus under milder conditions and they are capable of dealing with delayed self information. Tan et al. in (Tan and Liu [2013]) design a distributed protocol to actively compensate for the communication delay based on the adaptive control scheme and dynamic output feedback. Regarding the convergence rate of consensus under delays, the authors in (Bliman et al. [2008], Angeli and Bliman [2009], Nedic and Ozdaglar [2010]) provide bounds on the time required to reach consensus, which are explicit functions of the system parameters. Studying the model considering that a node may receive multiple messages from the same neighbor in one iteration, (Tsianos and Rabbat [2011, 2012]) reveals that the delays cannot slow down consensus by more than a polynomial factor under fixed finite delays and also exploits the convergence of consensus under time-varying delays. Chen et al. provide an upper bound of the convergence rate for discrete-time consensus with the switching topology and time-varying delays by developing a contractive-set method (Chen et al. [2016]).

Motivation: The aforementioned existing works mainly focus on the stability conditions and the upper bound analysis of the convergence rate of general consensus under time delays. However, few of them address other performances of discrete-time average consensus like the final value and the convergence rate. For example, in average consensus, the average of all nodes’ initial states is a critical concern, which should be considered when investigating the influence of time delays on consensus. Meanwhile, although it is intuitive that time delays

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will decrease the convergence rate of consensus, to the best of our knowledge, no work provides rigorous proof.

In this paper, apart from the stability analysis, the influence of uniform constant time delays on the final value and the convergence rate of discrete-time average consensus are also investigated. Uniform constant delays means that delays on all communication links are the same and constant, which is a simplified assumption. Yet, the results can be the explicit form, which can facilitate our future analysis on average consensus under time-varying delays. Besides that, through the buffering strategy, the time-varying delays can be viewed as constant ones (Lee and Spong [2006]), which means that when the communication environment is the same for all links, time-varying delays can be transformed to the uniform constant ones. The main contributions of this paper are summarized as follows.

1. We prove the stability of discrete-time average consensus under uniform constant time delays by analyzing the primitiveness of the augmented update matrix. It shows that average consensus is stable as long as the time delay is bounded.

2. We analyze the deviation of consensus value from the average and obtain the explicit expression, which reveals the relationship between the deviation and the system parameters. It is shown that the final value of the average consensus is robust to time delays if the communication topology is a regular graph.

3. By showing that the SLEM of the update matrix increases with the increment of time delays, we further provide rigorous proof for regular graph that the convergence rate of average consensus decreases under time delays.

The remainder of this paper is organized as follows. Section 2 models the network and formulates the problem. Main results are presented in Section 3. Section 4 verifies the proposed theoretical results through extensive simulations. Section 5 concludes the paper.

2. PROBLEM FORMULATION

2.1 Network Model

Consider an undirected connected network with \( n \) nodes which are indexed by \( 1, 2, \cdots, n \). A graph \( G = (V, E) \) is used to represent the topology of this network, where \( V \) denotes the set of nodes and \( E \subset V \times V \) denotes the set of communication links (edges). Assume that \((i, j) \in E \) if and only if \((i, j) \in E \), which means that the communication between node \( i \) and \( j \) is bidirectional. The neighbor set of node \( i \) is defined as \( N_i = \{j | (j, i) \in E, i \neq j, \forall j \in V \} \). The degree \( d_i \) of node \( i \) is the cardinality of \( N_i \), namely \( d_i = |N_i| \), and \( d_M = \max_{x \in V} d_i \). The adjacency matrix of the communication graph is denoted by \( A \) and the corresponding degree matrix whose diagonal elements are degrees of each node is denoted by \( D \). Assume that each node starts with a scalar state \( x_i(0) \) for \( i \in V \) and let \( x(t) = \{x_1(t), x_2(t), \cdots, x_n(t)\} \) be the state vector of all nodes at time \( t \).

2.2 Average Consensus Model

The dynamics of a general discrete-time consensus can be expressed as follows,

\[
x_i(t + 1) = w_{ii}x_i(t) + \sum_{j \in N_i} w_{ij}x_j(t), \quad i \in V,
\]

where \( w_{ii} \) and \( w_{ij} \) represent the weights. The matrix form of (1) is given as follows,

\[
x(t + 1) = Wx(t).
\]

An average consensus is achieved if,

\[
\lim_{t \to \infty} x_i(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(0).
\]

When \( W \) is a doubly stochastic matrix, average consensus can be achieved by (2). Hence, to guarantee average consensus, a classical stochastic weight model Xiao et al. [2005] is adopted in this paper, and the weight \( w_{ij} \) satisfies

\[
w_{ij} = \begin{cases} 
\frac{1}{\eta} & j \in N_i \\
1 - d_j/\eta & j = i \\
0 & j \notin N_i
\end{cases}.
\]

where \( \eta \) is assumed to satisfy \( \eta > d_M + 1 \). Let the Laplacian matrix \( L = D - A \), then the update matrix \( W \) can be related to the Laplacian matrix \( L \) as

\[
W = I - \frac{1}{\eta} L.
\]

With the delays between each pair of nodes, (1) will become

\[
x_i(t + 1) = w_{ii}x_i(t) + \sum_{j \in N_i} w_{ij}x_j(t - \tau), \quad i \in V.
\]

where \( \tau \) is the delays between neighboring nodes. These delays could result from the communication delays, the asynchronous clocks, or even attackers. Note that in discrete-time consensus, nodes update their states in a periodic way. Hence, the uniform constant time delays modeled in (4) does not mean that the real delays between the nodes are exactly the same and they just fall within the same update period of the nodes.

3. MAIN RESULTS

In this section, the performances of consensus algorithm (4), including the stability, the final value and the convergence rate, are investigated.

The dynamic (4) is rewritten by a common augmenting approach. Specifically, augmenting the state vector \( x(t) \) as \( y(t) = [x(t)^T, x(t - 1)^T, \cdots, x(t - n)^T]^T \), then we can write (4) in a matrix form as

\[
y(t) = \Phi(\tau)y(t - 1),
\]

where

\[
\Phi(\tau) = \begin{bmatrix}
W^d & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}_{(n+\tau)(n+\tau+1)}.
\]

The matrix \( W^d \) is constructed by replacing the diagonal of an \( n \times n \) identity matrix by the diagonal of \( W \), and \( W^d = W - W^d \). Hereafter, the dynamic equation (5) is the main concern and the performance of consensus under uniform constant time delays is investigated based on the properties of \( \Phi(\tau) \).

3.1 Stability Analysis

The stability of discrete-time consensus under time delays has been addressed extensively in the existing literatures. Here, a matrix theory-based approach is provided to prove that (5) will converge. Firstly, it is easy to verify that \( \Phi(\tau) \) is a row stochastic matrix, and then it is enough to show that \( \Phi(\tau) \) is primitive.
Theorem 1. \(\Phi(\tau)\) is a primitive matrix if the original graph \(G\) is an undirected connected graph, and then \(\Phi(\tau)\)' will converge.

Proof. The mathematical induction method is exploited here to prove the irreducibility of \(\Phi(\tau)\).
When \(\tau = 0\), \(\Phi(0) = W\), the conclusion holds naturally since \(G\) is undirected connected.
Assume that \(\Phi(\tau)\) is an irreducible matrix when \(\tau = k-1\), which also means that the associated graph \(G(k-1)\) (consisting of \(nk\) nodes) with \(\Phi(k-1)\) is an undirected connected graph.
For \(\tau = k\), compared with \(\Phi(k-1)\), there are \(n\) nodes labeled \([nk+1, nk+2, \ldots, n(k+1)]\) in the associated graph \(G(k)\) than that in \(G(k-1)\). These \(n\) nodes are inserted into graph \(G(k-1)\) by breaking down all the links between the nodes labeled \([1, 2, \ldots, n]\) and that labeled \([n(k-1)+1, n(k-1)+2, \ldots, nk]\).
Since \(G(k-1)\) is assumed to be strongly connected, \(G(k)\) is also strongly connected and the corresponding matrix \(\Phi(k)\) is irreducible.

Now that \(\Phi(\tau)\) is an irreducible matrix, combined with the fact that there are positive elements in the diagonal of \(\Phi(\tau)\), \(\Phi(\tau)\) is also primitive. Hence, \(\Phi(\tau)\)' will converge, which completes the proof.

Remark 1. The idea of breaking down links and inserting nodes could also be implemented to prove the irreducibility of the update matrix under nonuniform time delays.

Similar to the results in the existing works, consensus is robust to time delays with respect to the stability. However, the fact that time delays will degrade the performance of consensus regarding the final value and the convergence rate are almost ignored, and it is the main concern in the rest of this paper.

3.2 Final Value Analysis

It is observed that \(\Phi(\tau)\) is no longer a doubly stochastic matrix, which means that the dynamic process (5) cannot reach average consensus. In this subsection, the deviation of consensus value from the average is analyzed to show the influence of time delays. Let
\[
\Delta(\tau) = x_i(\tau) - \bar{x}
\]
be the deviation of consensus under time delays \(\tau\), where \(x_i(\tau)\) is the final value achieved by (4). The following theorem gives the explicit expression of \(\Delta(\tau)\).

Theorem 2. Due to the existence of time delays, the final value of consensus will deviate from the average of the initial values, and the deviation \(\Delta(\tau)\) satisfies
\[
\Delta(\tau) = \sum_{i=1}^{n} \frac{\tau n d_i - \sum_{j=1}^{n} d_j}{n(\eta + \tau \sum_{j=1}^{n} d_j)} x_i(0) . \tag{6}
\]

Proof. As \(\Phi(\tau)\) is still a row stochastic matrix, the right eigenvector corresponding to eigenvalue 1 is \(u = [1, 1, \ldots, 1]^T\) at \(\tau = 0\). The left eigenvector \(v\) of \(\Phi(\tau)\) corresponding to eigenvalue 1 is constructed as \(v = [v_1^T, v_2^T, \ldots, v_{\tau+1}^T]^T\), then
\[
v^T \Phi(\tau) = v^T ,
\]
which means that
\[
\begin{align*}
v_1^T W_d + v_2^T = v_1^T \\
v_2^T = v_1^T \\
\vdots \\
v_{\tau+1}^T = v_1^T
\end{align*}
\]  
By back substituting from the last equation of the equation set (7), one can get
\[
v_1^T W_d + v_1^T W_d = v_1^T W = \eta v_1^T ,
\]
which means that \(v_1^T\) is the left-hand eigenvector of the update matrix \(W\). Since \(W\) is a doubly stochastic matrix, \(v_1^T\) has the form
\[
v_1^T = k [1, 1, \ldots, 1]^T_{\text{loc1}},
\]
where \(k\) is a weighting factor.

According to equations (7),
\[
v_2^T = v_1^T = \cdots = v_{\tau+1}^T = v_1^T W_d = \frac{k}{\eta} [d_1, d_2, \ldots, d_n]^T .
\]
By normalizing \(v\), we can compute the weighting factor as
\[
k = \frac{\eta}{\eta n + \tau \sum_{j=1}^{n} d_j}.
\]
According to the limiting property of primitive matrices,
\[
\lim_{\tau \to \infty} \Phi(\tau)' = \frac{uv^T}{v_1^T u}
\]
we have
\[
\lim_{\tau \to \infty} y(t) = \lim_{\tau \to \infty} \Phi(\tau)' y(0) = uv^T y(\tau).
\]
If the nodes keep their states unchanged when they do not receive any states from their neighbors, i.e.,
\[
x_i(t) = x_i(0), \forall t < \tau, \forall i \in V ,
\]
\[
y(\tau) = [x(0)^T, x(0)^T, \cdots, x(0)^T]^T . \tag{7}
\]
Hence, the final value of consensus with time delays is,
\[
\lim_{\tau \to \infty} x_i(t) = uv^T x(\tau) = \sum_{j=1}^{\tau+1} v_j^T x_j(0), \forall i \in V .
\]

Then the deviation \(\Delta(\tau)\) from the average is
\[
\Delta(\tau) = \sum_{i=1}^{n} \frac{\kappa n d_i - \sum_{j=1}^{n} d_j}{n(\eta + \tau \sum_{j=1}^{n} d_j)} x_i(0) . 
\]

Remark 2. The method used in Theorem 2 could also be extended to the nonuniform constant time delays case, where the difference is that some part of the update matrix \(W\) will left multiply on \(v_{\tau+1}^T\) in the equation \(v_{\tau+1}^T = v_1^T, k \in [2, \tau]\) of the equations set (7).

When \(\tau \neq 0\), the deviation \(\Delta(\tau) \neq 0\) in most cases. However, if the topology of the network satisfies \(nd_i - \sum_{j=1}^{n} d_j = 0\), which means that graph \(G\) is a regular graph, the uniform time delays do not affect the final value of consensus.

To find out how \(\Delta(\tau)\) varies with \(\tau\), the difference between \(\Delta(\tau + 1)\) and \(\Delta(\tau)\) is obtained as follows,
\[
\Delta(\tau + 1) - \Delta(\tau) = \sum_{i=1}^{n} \frac{\eta (nd_i - \sum_{j=1}^{n} d_j)}{(n + \tau \sum_{j=1}^{n} d_j)(n + (\tau + 1) \sum_{j=1}^{n} d_j)} x_i(0). \tag{8}
\]

It is observed from (8) that how \(\Delta(\tau)\) varies with \(\tau\) is determined by the topology of the network and the initial states distribution.
For a specific combination of the system parameters including \(d_i\) and \(x_i(0)\), \(\Delta(\tau)\) is monotonic with \(\tau\) and the rate of change decreases.
3.3 Convergence Rate Analysis

Intuitively speaking, since the nodes will use outdated neighboring states repeatedly due to the existence of time delays, the convergence rate of consensus will slow down. However, to the best of our knowledge, no works provide guarantees on this fact. In this subsection, a solid proof for a special topology is given by comparing the SLEM of \( W \) with that of \( \Phi(\tau) \), which validates the intuition in some cases.

Firstly, the characteristic equation of \( \Phi(\tau) \) is computed by

\[
|I - \Phi(\tau)| = \begin{vmatrix}
\lambda - W^d & 0 & \cdots & 0 \\
-1 & \lambda - W^d & 0 & 0 \\
0 & -1 & \lambda - W^d & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda - W^d \\
\end{vmatrix}.
\]

For the right hand determinant of the (9), an operation is defined as follows. For each row \( i \) (in block sense, the same hereafter), multiplying row \( i \) by \( \lambda \) and adding it to the \((i - 1)\)-th row, then eliminating the \( i \)-th element in the last column by multiplying the \((i - 1)\)-th column by a coefficient and adding it to the last column. By repeating the operation from the last row to the second row in order, then, we can obtain

\[
|I - \Phi(\tau)| = \\
|\lambda - W^d & 0 & \cdots & 0 \\
-1 & \lambda - W^d & 0 & 0 \\
0 & -1 & \lambda - W^d & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda - W^d - W^d_{i-1} \\
\]

It is difficult to analyze the eigenstructure of \( |I - \Phi(\tau)| \) for general topologies since the determinant cannot be expanded. However, when the original associated graph \( G \) is a regular graph, we have \( W^d = (I - \frac{d}{\eta})I \) where \( d \) is the degree of each node. To these special topologies, \( |I - \Phi(\tau)| \) could be expressed as a tractable polynomial and a rigorous proof can be obtained.

For regular graphs, the characteristic equation of \( \Phi(\tau) \) can be rewritten as,

\[
|I - \Phi(\tau)| = |\lambda^{\tau+1} - (1 - \frac{d}{\eta})\lambda^{\tau}I - W^d| = 0,
\]

As \( W^d \) is a symmetric matrix, there exists an orthogonal matrix \( P \) such that \( P^{-1}W^dP = \Lambda \), then,

\[
|I - \Phi(\tau)| = |P^{-1}||\lambda^{\tau+1} - (1 - \frac{d}{\eta})\lambda^{\tau}I - W^d||P|
\]

\[
= |\lambda^{\tau+1} - (1 - \frac{d}{\eta})\lambda^{\tau}I - \Lambda|,
\]

Theorem 3. In regular graphs, the SLEM of \( \Phi(\tau) \) is larger than or equal to that of \( W \), which means that the convergence rate of consensus decreases due to time delays.

Proof. Denote the \( i \)-th eigenvalue of \( \Lambda \) and hence \( W^d \) as \( \mu_i \).

Then all eigenvalues of \( \Phi(\tau) \) are constituted by the solutions of the following equations,

\[
\lambda^{\tau+1} - (1 - \frac{d}{\eta})\lambda^{\tau} = 0.
\]

The \( i \)-th eigenvalue of \( W \) can be expressed as \( v_i = \mu_i + (1 - \frac{d}{\eta}) \), and the SLEM \( |\nu| = |\mu^\tau + (1 - \frac{d}{\eta})| \) governs the rate of consensus without delays. Hereafter, it is proven that there exists at least one solution \( \lambda^\tau \) of (11) such that \( |\lambda^\tau| > |\nu| \). Two cases are considered in the following, i.e., \( \nu^\tau = 0 \) and \( \nu^\tau > 0 \).

When \( \nu^\tau = 0 \), which means that \( \mu^\tau = -(1 - \frac{d}{\eta}) \), (11) can be modified as

\[
\lambda^{\tau+1} - (1 - \frac{d}{\eta})\lambda^{\tau} = 0.
\]

Let \( H_0(z) = z^{\tau+1} - z^\tau + (1 - \frac{d}{\eta}) \), and \( \tilde{H}_0(z) = z^{\tau+1}H_0(z^{\tau}) = \frac{1}{1 - (\frac{d}{\eta})}z^{\tau+1} - z + 1, \) then,

\[
\tilde{H}_0(z) / H_0(z) = \frac{1}{1 - (\frac{d}{\eta})} + \tilde{H}_1(z).
\]

Since \( \frac{1}{1 - (\frac{d}{\eta})} > 1 \), according to the modified Schur-Cohn test Tretter [1976], \( H_0(z) \) has at least one solution located outside of the unit circle, which also means that there is at least one solution \( \lambda^\tau \) such that \( |\lambda^\tau| \geq (1 - \frac{d}{\eta}) > 0 = \nu^\tau \).

When \( \nu^\tau > 0 \), which means that \( \mu^\tau = -(1 - \frac{d}{\eta}) > 0 \), \( \mu^\tau \) must satisfy \( \mu^\tau \geq 0 \). As the diagonals of \( W^d \) sum to 0, \( \sum_{i=1}^n \mu_i = 0 \) holds and \( |\mu| \leq 1 - \frac{d}{\eta} \). Hence \( \nu^\tau \) must correspond to a \( \mu^\tau \) which is larger than or equal to 0. Then, (11) can be modified as

\[
(\frac{1}{\nu^\tau})^{\tau+1} - (1 - \frac{d}{\eta})\nu^\tau - \frac{\mu^\tau}{(\nu^\tau)^{\tau+1}} = 0.
\]

Let \( H_0(z) = z^{\tau+1} - (\frac{1}{\nu^\tau})z^\tau - (\frac{\mu^\tau}{(\nu^\tau)^{\tau+1}}) \), and since \( \nu^\tau \leq 1 \), it is easy to verify that

\[
H_0(1) = 1 - (1 - \frac{d}{\eta})\nu^\tau - \frac{\mu^\tau}{(\nu^\tau)^{\tau+1}} \leq 1 - (1 - \frac{d}{\eta}) - \frac{\mu^\tau}{\nu^\tau} = 0,
\]

which again violates the modified Schur-Cohn test. Hence, there is at least one solution \( \lambda^\tau \) such that \( |\lambda^\tau| \geq |\nu| \).

To sum up, the SLEM of \( \Phi(\tau) \) is larger than that of \( W \), which means that the convergence rate of consensus decreases due to the existence of time delays.

In Theorem 3, it is proven that the SLEM of \( \Phi(\tau) \) is larger than or equal to that of \( W \). However, it is not clear how the SLEM of \( \Phi(\tau) \) varies with \( \tau \). The theorem below proves that the SLEM of \( \Phi(\tau) \) increases with the time delays, which means that the larger the time delays, the slower the average consensus converges.

Theorem 4. In regular graphs, the SLEM increases with time delays.

Proof. For \( \tau = k - 1 \), the SLEM \( \lambda^\tau \) of \( \Phi(k - 1) \) satisfies

\[
(\lambda^\tau)^{k-1} - (1 - \frac{d}{\eta})(\lambda^\tau)^{k-1} - \mu_i = 0.
\]

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For $\tau = k$, the eigenvalue of $\Phi(k)$ corresponding to the same $\mu_i$ satisfies
\[
\lambda^{k+1} - (1 - \frac{d}{\eta})\lambda^k - \mu_i = 0.
\]
Let $H_0(z) = z^{k+1} - (1 - \frac{d}{\eta})z^k - \mu_i$ ($\lambda^* = 0$ is a trivial case) and substituting $\mu_i = (\lambda^*)^k - (1 - \frac{d}{\eta})(\lambda^*)^{k-1}$ into $H_0(z)$,
\[
H_0(z) = z^{k+1} - \frac{d}{\eta}z^k - \frac{1}{\lambda^*} + \frac{1}{(\lambda^*)^2}.
\]
Then,
\[
H_0(1) = 1 - \frac{d}{\eta} - \frac{1}{\lambda^*} + \frac{1}{(\lambda^*)^2} = \frac{(\lambda^*)^2 - (2 - \frac{d}{\eta}) + 1 - \frac{d}{\eta}}{(\lambda^*)^2} = \frac{(\lambda^* - (1 - \frac{d}{\eta}))(\lambda^* - 1)}{(\lambda^*)^2}.
\]
It is shown in Theorem 3 that $\lambda^* \geq 1 - \frac{d}{\eta}$ (in both cases of $\nu^* = 0$ or $\nu^* > 0$). Therefore, $H_0(1) \leq 0$, which violates the modified Schur-Cohn test. Therefore, the SLEM of $\Phi(k)$ is larger than that of $\Phi(k - 1)$, which completes the proof.

**Remark 3.** The convergence rate analysis given in this subsection is only valid for regular graph. However, the result should also be applied to the general topologies by intuition, which is left as our future work.

By Theorem 3 and 4, the convergence rate of consensus is also negatively correlated with delays, which verifies our intuition.

### 4. SIMULATION VERIFICATION

In this section, extensive simulations are conducted to verify the obtained theoretical results regarding the stability, the final value and the convergence rate of consensus under time delays.

#### 4.1 Simulation Setup

Consider a network consisting of $n = 12$ nodes, which are randomly distributed on a $20m \times 20m$ square region. The communication range of each node is $4m$. The initial states of the nodes are randomly selected from $[0, 10]$. The maximum distance between the nodes' states, i.e.,
\[
\delta(t) = \max_{i \in V} x_i(t) - \min_{i \in V} x_i(t),
\]
is utilized to characterize the convergence of consensus. The parameter $\eta$ in (3) is set to be $n$.

#### 4.2 Convergence

![Fig. 1. Stability under different time delays](image)

Under different time delays, i.e., $\tau = 0, 5, 10, 15$, the convergence curves are depicted in Fig. 1. It is observed that consensus still converges under uniform constant time delays. However, when the time delays are relatively large, the convergence curve oscillates at the beginning of the iteration, which is induced by repeatedly using outdated initial states. From these curves, it is also noticed that consensus converges slower with the increment of the time delays even if the topology of the network is not a regular graph, which again supports the potential generalization of the results in Theorem 3 and 4.

#### 4.3 Deviation

![Fig. 2. Deviation in non-regular graph](image)

**Fig. 2.** Deviation in non-regular graph

As claimed in Theorem 2, the final value of consensus will deviate from the average under time delays in most topologies. For $\tau = 5$, Fig. 2(a) clearly shows the final value deviation in a non-regular graph and the deviation value coincides with the result computed by (6). For the same system parameters setup, i.e., the network topology and initial states distribution, how the deviation $\Delta(\tau)$ varies with $\tau$ is illustrated in Fig. 2(b).

In contrast, as shown in Fig. 3, in a 2-regular graph ($d = 2$ for all nodes) with the same nodes’ initial states distribution, consensus still converges to the average under uniform constant time delays. Hence, if the underlying topology of the network is a regular graph, consensus is robust to time delays regarding both the stability and the final value.

![Fig. 3. Deviation in regular graph](image)

**Fig. 3.** Deviation in regular graph

#### 4.4 Convergence Rate

In a 2-regular graph, the variation of the SLEM of $\Phi(\tau)$ with $\tau$ is shown in Fig. 4. It is observed that when $\tau > 0$, the SLEM of
The SLEM of $\Phi(\tau)$ is larger than that of $W$ ($\tau = 0$), which verifies the result of Theorem 3. Meanwhile, the SLEM of $\Phi(\tau)$ increases with the increment of the time delays $\tau$ as claimed in Theorem 4.

This paper investigated how the performance of discrete-time average consensus is affected by uniform constant time delays. We first proved that consensus is still stable under bounded time delays via a matrix theory-based approach. Then, by comparing the final value with the real average, we obtain an explicit expression of the deviation caused by the time delays. It shows that for the regular graph, average consensus is preserved under time delays. Finally, we prove that for regular graphs, the convergence rate of consensus decreases with the increment of time delays. Extensive simulations validate the obtained theoretical results. For future research directions, we should extend the convergence rate result from regular graphs to general topologies, and quantitative analysis and time-varying delays should also be addressed.

**REFERENCES**


